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The Effect of Small Phase Errors Upon Transmission Between Confocal Apertures

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The effect of small, periodic phase errors upon transmission between two coaxial, circularly symmetric apertures is considered when the aperture phase distributions are confocal and the amplitude distributions are gaussian. The results are applicable to loss calculations in beam waveguide systems with imperfect lenses. When the periods of the phase errors are less than one-half the aperture radii, the total loss is approximately $\frac{1}{2}(\beta_1^2 + \beta_2^2)$, *where* β_1, β_2 *are the peak phase errors (in radians) on the apertures. Phase errors with periods greater than the aperture diameters are found to cause comparatively little transmission loss.*

I. INTRODUCTION

The use of beam waveguide[1] systems for the transmission of information,[2] or for the transmission of power,[3] necessitates the design of lenses (or cylindrical reflectors[4]) as focusing elements. In the design of these elements, it is desirable to estimate the degradation in performance caused by surface profile errors. Such degradation results in transmission loss and, in a communications system, will contribute to interference. Typically, the profile errors are associated with machining operations and, for lenses with circular symmetry, these errors are frequently circularly symmetric. The principal effect of the errors is to impart small, circularly symmetric phase perturbations to the field distribution adjacent to the lenses. The purpose of this paper is to calculate the reduction in transmission, caused by phase errors of this type, in a simple system comprising two coaxial, circular apertures as shown in Fig. 1. The field distributions on the apertures may represent the fields in the aperture planes of two antennas or the fields on adjacent lenses in a beam waveguide system.

In the absence of phase errors the transmission between coaxial apertures has been extensively studied by Kay,[5] Borgiotti,[6] Heurtley,[7] and others, with the principal objective of determining that aperture

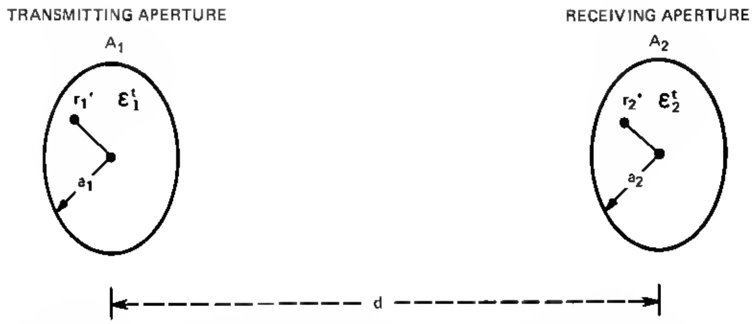

Fig. 1—Coaxial, circular apertures with confocal field distributions $\mathcal{E}_1^t(r_1')$ and $\mathcal{E}_2^t(r_2')$.

field distribution which maximizes the transmission. When the aperture separation is greater than some minimum, it has been found that this field distribution corresponds to that of the lowest order mode in an open, confocal resonator.[8] The phase distribution appropriate to this mode is obtained when the phase fronts on the apertures are confocal, i.e., are spherical with the center of curvature at the center of the other aperture. The appropriate amplitude distribution is well approximated[9] by a gaussian curve. For this (optimum) distribution, the transmission between the apertures can attain surprisingly high values even when the apertures are separated by many aperture diameters (see Refs. 5, 6, and 7). Although the effect of periodic and random phase errors upon antenna gain and side lobe level has been investigated by several authors,[10,11] little information appears to be available regarding transmission between two apertures when each has phase errors. In the case of transmission between two reflector antennas, Chu[12] has obtained an upper bound for the loss resulting from those phase errors produced by displacement of the feeds from the reflector foci. Yoneyama and Nishida[13] have considered transmission through a two-dimensional, confocal beam waveguide consisting of lenses with random phase errors. We compare their results to those of the present study later in this paper.

In the following section the total transmission loss in a confocal system, with small phase errors on apertures with arbitrary amplitude distributions, is expressed in terms of the losses associated with each aperture when the other is free from phase errors. In the next section explicit expressions are derived, for two cases of practical interest, when the phase errors are sinusoidal and when the aperture amplitude distributions are gaussian. These expressions are then discussed and compared with results from the literature.

II. TRANSMISSION BETWEEN CONFOCAL APERTURES WITH PHASE ERRORS

Consider the circular apertures A_1, A_2 of radius a_1, a_2 which are separated by a distance $d > a_1, a_2$, as in Fig. 1. It is assumed that the apertures are focused at each other such that the tangential electric fields in the apertures, when each is transmitting in the absence of the other, have the quadratic phase variation

$$\mathcal{E}_i^t(r_i') = E_i(r_i') \exp\left(j\,\frac{kr_i'^2}{2d}\right), \qquad i = 1, 2, \tag{1}$$

where the r_i' are radial coordinates in the apertures. In the absence of phase errors the $E_i(r_i')$ are real. If interaction is neglected, the transmission between the apertures is readily found[7,12] from the results of Hu[14] and Kay[5]:

$$T = \frac{1}{D}\left|\int_0^1 \int_0^1 F_{12}dr_1dr_2\right|^2, \tag{2}$$

where

$$F_{12} = E_1(r_1)E_2(r_2)J_0(nr_1r_2)r_1r_2 \tag{3}$$

and

$$D = \frac{1}{n^2}\left[\int_0^1 |E_1(r_1)|^2 r_1dr_1 \int_0^1 |E_2(r_2)|^2 r_2dr_2\right]. \tag{4}$$

In these expressions the r_i are normalized so that $r_i = r_i'/a_i$. The Fresnel number $n = ka_1a_2/d$, with k the wave number, and J_0 is the Bessel function of order zero. In the special case when the aperture separation is much greater than the aperture diameters, we see that $n \ll 1$ and that the aperture phases are uniform, i.e., $\mathcal{E}_i^t(r_i) = E_i(r_i)$. Substituting the small argument approximation for the Bessel function, $x \ll 1$, $J_0(x) \approx 1$, eq. (2) then reduces to the familiar Friis transmission formula[15]

$$T = \frac{A_1^e A_2^e}{(\lambda d)^2}, \qquad n \ll 1. \tag{5}$$

The effective aperture areas A_i^e are defined by

$$A_i^e = 2\pi a_i^2 \frac{\left|\int_0^1 \mathcal{E}_i^t(r)rdr\right|^2}{\int_0^1 |\mathcal{E}_i^t(r)|^2 rdr}, \qquad i = 1, 2, \tag{6}$$

e.g., in the case of uniform illumination, $\mathcal{E}_i^t(r) = 1$ and $A_i^e = \pi a_i^2$. The far-field transmission, T in (5), is also expressible in terms of the gains (G) of the apertures A_1 and A_2:

$$T = G_1G_2\left(\frac{\lambda}{4\pi d}\right)^2, \qquad n \ll 1, \tag{7}$$

where

$$G_i = \frac{4\pi A_i^2}{\lambda^2}, \qquad i = 1, 2. \tag{8}$$

Returning now to the discussion of phase errors, suppose that the phases in the apertures depart from the ideal (confocal) distributions by amounts $\phi_1(r_1)$ in A_1 and $\phi_2(r_2)$ in A_2. The transmission T_{12} between the apertures in the presence of these phase errors is, from (2),

$$T_{12} = \frac{1}{D}\left|\int_0^1\int_0^1 F_{12}\exp\left[j(\phi_1+\phi_2)\right]dr_1dr_2\right|^2, \tag{9}$$

where the $\phi_i(r_i)$ are abbreviated to ϕ_i. T_{12} is expressible as

$$T_{12} = T_0 - \Delta T_{12}, \tag{10}$$

where T_0 is the transmission in the absence of phase errors and ΔT_{12} is the loss resulting from the phase errors. Let

$$T_i = T_0 - \Delta T_i, \qquad i = 1, 2 \tag{11}$$

be the transmission between the apertures when there is a phase error on only the aperture A_i. From Appendix A we then find

$$\Delta T_{12} \approx \Delta T_1 + \Delta T_2 + R, \tag{12}$$

where

$$\Delta T_i = \frac{1}{D}\left[\int_0^1\int_0^1 F_{12}dr_1dr_2 \int_0^1\int_0^1 F_{12}\phi_i^2 dr_1dr_2 - \left\{\int_0^1\int_0^1 F_{12}\phi_i dr_1dr_2\right\}^2\right], \quad i = 1, 2, \tag{13}$$

$$R = \frac{2}{D}\left[\int_0^1\int_0^1 F_{12}dr_1dr_2 \int_0^1\int_0^1 F_{12}\phi_1\phi_2 dr_1dr_2 - \int_0^1\int_0^1 F_{12}\phi_1 dr_1dr_2 \int_0^1\int_0^1 F_{12}\phi_2 dr_1dr_2\right]. \tag{14}$$

Equation (12) states that the total loss incurred by (small) phase errors $\phi_1(r_1)$ and $\phi_2(r_2)$ on confocal apertures A_1, A_2 is approximately equal to the sum of the losses associated with each aperture when the other is free of phase errors together with the term R. In the next section we evaluate the expressions for ΔT_i and R when the aperture amplitude distributions are gaussian.

III. GAUSSIAN APERTURES

In the case of gaussian amplitude distributions, $E_i(r_i) = \exp(-\alpha_i r_i^2)$. To simplify the analysis we assume the α_i to be sufficiently large that the upper limits in the integrals may be extended to infinity. The

transmission T_0 in the absence of phase errors may now be obtained from (2), (3), and (4) by noting[16]

$$\int_0^\infty \exp(-\alpha_2 r_2^2) J_0(n r_1 r_2) r_2 dr_2 = \frac{1}{2\alpha_2} \exp\left(-\frac{n^2 r_1^2}{4\alpha_2}\right) \tag{15}$$

to give

$$T_0 = \frac{16 n^2 \alpha_1 \alpha_2}{(n^2 + 4\alpha_1\alpha_2)^2}, \qquad \exp(-\alpha_i) \ll 1, \qquad i = 1, 2. \tag{16}$$

Apart from differences in notation, this expression is identical to the corresponding result obtained by Kogelnik[17] for the coupling of (fundamental) gaussian modes. We find

$$T_0 = 1 \qquad \text{when} \qquad n = 2\sqrt{\alpha_1 \alpha_2}, \tag{17}$$

i.e., within the approximation $\exp(-\alpha_i) \ll 1$, there are optimum amplitude distributions that will ensure complete power transfer between the apertures for a given n. A detailed analysis,[9] or numerical integration, indicates this to be a satisfactory approximation when $\alpha_i \gtrsim 2.3$, $i = 1, 2$. For example, when $\alpha_1 = \alpha_2 = 2.36$, the exact[9] results for identical apertures are $T_0 = 0.9931$, $n = 5.00$, and the approximate results are $T_0 = 1.00$, $n = 4.72$. From (6), the effective aperture area, A_i^e, of a gaussian aperture is

$$A_i^e = \frac{2\pi a_i^2}{\alpha_i}, \qquad i = 1, 2. \tag{18}$$

As expected from physical considerations A_i^e decreases as α_i increases, i.e., as the aperture field becomes more concentrated about the aperture center.

In the case of transmission with circularly symmetric, periodic phase errors, we take

$$\phi_i(r_i) = \beta_i \cos(\gamma_i r_i), \qquad i = 1, 2,$$

where (19)

$$\beta_i = k\delta_i, \qquad \gamma_i = \frac{2\pi a_i}{l_i}.$$

β_i is the peak value of the error in radians (with δ_i the peak profile deviation) and l_i is the period of the error. For errors of period much greater than the circumference of the apertures, $\gamma_i \ll 1$ and then, in (9), $\phi_i(r_i) \approx \beta_i$, $i = 1, 2$. It follows, therefore, that $T_{12} = T_0$, i.e., to this order of approximation, small, slowly varying, circularly symmetric phase errors do not affect transmission between the apertures. In the general case of small errors, the transmission loss ΔT_{12} for gaussian apertures is found from (12) with (13) and (14) by

substituting for the $E_i(r_i)$ and $\phi_i(r_i)$. From Appendix B, we have

$$\frac{\Delta T_i}{T_0} = \beta_i^2\gamma_i'[2\mathfrak{D}(\gamma_i'/2) - \mathfrak{D}(\gamma_i') - \gamma_i'\mathfrak{D}^2(\gamma_i'/2)], \qquad i = 1, 2, \tag{20}$$

where

$$\gamma_i' = \gamma_i\sqrt{\frac{4\alpha_j}{n^2 + 4\alpha_1\alpha_2}} \tag{21}$$

and

$$\mathfrak{D}(x) = \exp(-x^2)\int_0^x \exp(\tau^2)d\tau \tag{22}$$

is the (tabulated) Dawson integral.[18] The index $j = \{{2 \atop 1}\}$ when $i = \{{1 \atop 2}\}$.

The term R in (14) may be evaluated approximately in two cases of practical interest. In the first of these, the apertures are sufficiently far apart that $n \ll 1$ so that $J_0(nr_1r_2) \approx 1$ in (3). F_{12} is then separable in functions of r_1 and r_2 and hence, from (14), $R = 0$. For this case,

$$\Delta T_{12} \approx \Delta T_1 + \Delta T_2, \tag{23}$$

i.e., the total loss is approximately the sum of the losses associated with each aperture when the other aperture is free of phase errors. The total loss is given by (20) and (23) in which γ_i' simplifies to

$$\gamma_i' \approx \frac{\gamma_i}{\sqrt{\alpha_i}}, \qquad n \ll 1. \tag{24}$$

It is noted from (7) and (11) that

$$\frac{\Delta T_i}{T_0} = \frac{\Delta G_i'}{G_i}, \qquad n \ll 1, \quad i = 1, 2, \tag{25}$$

where $G_i' = G_i - \Delta G_i'$ is the gain of aperture A_i with the phase error ϕ_i. Hence, (20) with (24) gives the fractional reduction in gain of the aperture A_i resulting from a (small) periodic phase error.

The second case of practical interest arises when the amplitude distributions on the apertures are optimized in accordance with (17) such that, in the absence of phase errors, the transmission is unity. From Appendix C, the term R in (12) is negligible in this case provided $\gamma_1, \gamma_2 \gg n = 2\sqrt{\alpha_1\alpha_2}$, i.e., the periods ($l_i$) of the phase errors satisfy

$$l_1 \ll \frac{\lambda d}{a_2} \qquad \text{and} \qquad l_2 \ll \frac{\lambda d}{a_1}. \tag{26}$$

The transmission loss, ΔT_{12}, between the apertures is then the sum of the losses associated with each aperture as given by (23). This result

implies that the transmission through a sequence of confocal lenses, each with small phase errors of period satisfying (26), may be obtained by calculating the transmissions associated with each lens in the absence of pbase errors on the other lenses. Furthermore, (23) indicates that it is not possible to compensate for sucb pbase errors on one lens by introducing phase variations on an adjacent lens. When (26) is satisfied, the Dawson integrals in (20) may be replaced by the first terms of the asymptotic expansion (44) to give

$$\Delta T_i \approx \tfrac{1}{2}\beta_i^2, \qquad i = 1, 2. \tag{27}$$

It is of interest to note the physical significance of the condition (26) for the validity of the approximate forms (23) and (27). As expected from the theory of diffraction gratings, a circularly symmetric, periodic pbase perturbation on a circular aperture generates[19] two additional side lobes in the aperture radiation pattern. If the period of the phase perturbation is l, then the two side lobes are symmetrically located about the main beam at an angle $\theta = \sin^{-1}(\lambda/l)$. Consider now the two apertures of Fig. 1 with phase errors of period l_1 in A_1 and l_2 in A_2. If the apertures are sufficiently far apart the side lobes, due to the phase error l_1 in A_1, will not intercept A_2 provided $\sin^{-1}(\lambda/l_1) \gg \tan^{-1}(a_2/d)$, i.e., for small angles, $l_1 \ll \lambda d/a_2$. Similarly, the main beam of A_1 will not couple energy to the side lobes of A_2 provided $l_2 \ll \lambda d/a_1$. The condition (26) implies, therefore, that energy is coupled from A_1 to A_2 via the main beams alone.

Figure 2 shows the transmission, as a function of $\gamma = \gamma_1 = \gamma_2$, between two identical apertures as obtained by numerical integration of (9) with (19), and as obtained from the approximate result (23) with (27). The upper curve in the figure applies for $\alpha = 4$, $n = 8$, $\beta = 0.36$ and the lower curve for $\alpha = 2.36$, $n = 5$, $\beta = 0.18$. In the ahsence of phase errors $T_0 = 1$ for these (optimum) distrihutions. The dashed lines correspond to the approximation (23) with (27). As anticipated earlier, the transmission is essentially unaffected by phase errors of large period, e.g., when $\gamma \lesssim n/2$, i.e., $l \gtrsim 2\lambda d/a$. The approximate form (23) with (27) is seen to be within about 1 percent of the exact result when $\gamma \gtrsim 2n$, i.e., $l \lesssim \lambda d/2a$. As an illustrative example, consider a beam waveguide system of the type described hy Arnaud and Ruscio[2] with $\lambda = 3 \times 10^{-3}m$, $d \approx 80m$, $a \approx 0.5m$. The parameters of this system correspond to those of the lower curve in Fig. 2. Substitution shows that small, circularly symmetric phase errors of period $l \gtrsim 2a$ on the lenses will cause negligible transmission loss, and that tbe approximation (23) with (27) is applicable for phase errors of period $l \lesssim a/2$.

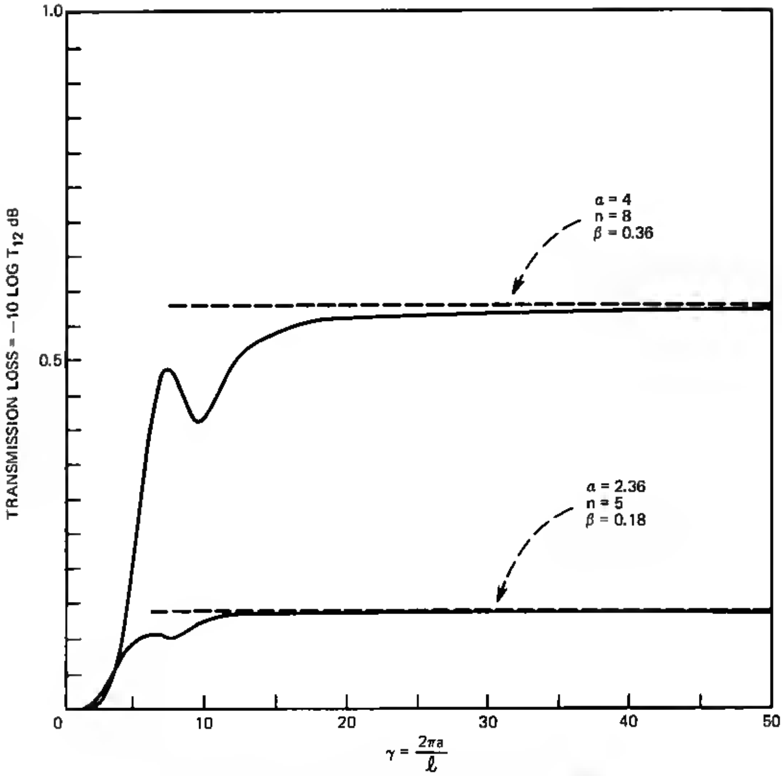

Fig. 2—Dependence on γ of phase error loss.

IV. COMPARISON WITH PREVIOUS RESULTS

To conclude this discussion on the effects of phase errors, we hriefly compare the preceding results with the work of others. An expression for the gain (G') of an aperture with small, periodic phase errors was given in (25). Consider the special case in which the period of the phase error is much less than the dimensions of the effective aperture, i.e., $l \ll \sqrt{A^e}$. From (18) and (19), this implies $\gamma \gg \sqrt{\alpha}$ so that the Dawson integrals in (20) with (24) may he replaced hy the large argument form (44) to give, with (25),

$$\frac{G'}{G} \approx 1 - \tfrac{1}{2}\beta^2, \tag{28}$$

where G is the gain in the ahsence of phase errors. Since this result depends only upon the magnitude β of the phase error, it is anticipated that it may apply to random phase errors with correlation lengths that are small compared with the dimensions of the effective aperture. Ruze[11] has examined the reduction in aperture gain caused by such

random errors and we compare (28) with his results. In the particular case of a sinusoidal surface error of rms value ϵ on a paraholic reflector antenna, we have $\beta = k\delta = 2\sqrt{2}k\epsilon$. From (28), the gain with this small phase error is

$$\frac{G'}{G} \approx \exp\left[-\left(\frac{4\pi\epsilon}{\lambda}\right)^2\right]. \tag{29}$$

This expression, derived here for a sinusoidal phase error, is identical to that ohtained hy Ruze in the case of a random error. As noted in Section I, Yoneyama and Nishida[13] have examined the effect of random phase errors on lenses in a two-dimensional, confocal, heam waveguide system. Their approach is hased on the concept of a statistical heam mode and this leads to a description of the field distrihution, and transmission loss, in terms of an integral equation. A computer was used to solve the integral equation hy numerical iteration from the solution in the ahsence of phase errors. It is interesting to find that the conclusions of their study, of a two-dimensional system with random errors, are similar to those ohtained here for transmission hetween circular apertures with periodic phase errors. In particular, it was found that the transmission was not appreciahly affected hy phase errors with large correlation lengths and that the loss for a given error tended to a constant value for increasingly small correlation lengths.

V. CONCLUSIONS

We have examined the effect of small, periodic, radial phase errors upon transmission hetween two coaxial, circularly symmetric apertures with confocal phase distrihutions. Two cases of practical interest have heen considered when the amplitude distrihutions on the apertures are gaussian. In the first of these the apertures are widely separated with phase errors of arhitrary period. The total loss is then the sum of the losses associated with each aperture and is given in terms of tahulated Dawson integrals. This result reduces to a known form when the periods of the phase errors are sufficiently small. The second case of interest applies to transmission through a heam waveguide system with imperfect lenses. When the periods (l_i) of the phase errors on the apertures satisfy $l_i \lesssim a_i/2$, ($i = 1, 2$), where a_i is the aperture radius, the total loss resulting from phase errors is approximately $\frac{1}{2}(\beta_1^2 + \beta_2^2)$, where β_1, β_2 are the peak phase errors in radians on the two apertures. A comparison, hased on numerical integration, shows this to he within ahout 1 percent of the exact result in a typical case. Phase errors with periods $l_i \gtrsim 2a_i$ ($i = 1, 2$) have comparatively little effect upon transmission.

VI. ACKNOWLEDGMENTS

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APPENDIX A

Derivation of (12)

For small phase errors, $(\phi_1 + \phi_2) \ll 1$ and the exponential in (9) may be expanded to second order. Recalling that the E_i are real, i.e., F_{12} is real, we then find

$$T_{12} \approx \frac{1}{D}\left[\left\{\int_0^1 \int_0^1 F_{12}[1 - \tfrac{1}{2}(\phi_1 + \phi_2)^2]dr_1dr_2\right\}^2 + \left\{\int_0^1 \int_0^1 F_{12}(\phi_1 + \phi_2)dr_1dr_2\right\}^2\right]. \quad (30)$$

Expanding the first bracket and noting that

$$\frac{1}{D}\left\{\int_0^1 \int_0^1 F_{12}(\phi_1 + \phi_2)^2dr_1dr_2\right\}^2 \leqq (\phi_1 + \phi_2)^4_{\max}T_0, \quad (31)$$

where $(\phi_1 + \phi_2)_{\max}$ is the maximum value of $(\phi_1 + \phi_2)$, we have, to second order in ϕ_1, ϕ_2,

$$T_{12} = T_0 - \Delta T_{12}, \quad (32)$$

where

$$\Delta T_{12} \approx \frac{1}{D}\left[\int_0^1 \int_0^1 F_{12}dr_1dr_2 \int_0^1 \int_0^1 F_{12}(\phi_1 + \phi_2)^2dr_1dr_2 - \left\{\int_0^1 \int_0^1 F_{12}(\phi_1 + \phi_2)dr_1dr_2\right\}^2\right]. \quad (33)$$

Expanding the brackets gives (12) with (13) and (14).

APPENDIX B

Derivation of (20)

Substituting $E_i(r_i) = \exp(-\alpha_i r_i^2)$, $\phi_i(r_i) = \beta_i \cos(\gamma_i r_i)$ $(i = 1, 2)$ into (13) with (3) and (4) gives, with (15),

$$\frac{\Delta T_i}{T_0} = 2\eta\beta_i^2[I_1(\gamma_i) - 2\eta I_2^2(\gamma_i)] \qquad i = 1, 2, \quad (34)$$

where

$$I_1(\gamma_i) = \int_0^\infty \exp(-\eta r^2)\cos^2(\gamma_i r)rdr, \quad (35)$$

$$I_2(\gamma_i) = \int_0^\infty \exp(-\eta r^2)\cos(\gamma_i r)rdr \quad (36)$$

and

$$\eta = \frac{1}{4\alpha_j}(n^2 + 4\alpha_1\alpha_2), \tag{37}$$

with $j = \{^2_1\}$ when $i = \{^1_2\}$. Expanding $\cos^2(\gamma_i r)$ and integrating:

$$I_1(\gamma_i) = \frac{1}{4\eta} + \tfrac{1}{2}I_2(2\gamma_i). \tag{38}$$

Integrating by parts,

$$I_2(\gamma_i) = \frac{1}{2\eta}\left[1 - \gamma_i \int_0^\infty \exp(-\eta r^2)\sin(\gamma_i r)dr\right], \tag{39}$$

which is expressible[18] in terms of the (tabulated) Dawson integral, i.e.,

$$I_2(\gamma_i) = \frac{1}{2\eta}\left[1 - \frac{\gamma_i}{\sqrt{\eta}}\mathfrak{D}\left(\frac{\gamma_i}{2\sqrt{\eta}}\right)\right], \tag{40}$$

where

$$\mathfrak{D}(x) = \exp(-x^2)\int_0^x \exp(\tau^2)d\tau. \tag{41}$$

From (34), (38), and (40), we then obtain (20) and from (19) and (37) we obtain (21).

APPENDIX C

Approximate Evaluation of R in (14)

We derive an approximate expression for R when $\gamma_1, \gamma_2 \gg n$. It is assumed that the amplitude distributions on the apertures are optimized such that $T_0 = 1$ with $n = 2\sqrt{\alpha_1\alpha_2}$, where $\alpha_1, \alpha_2 \geqq 2.3$. Consider the integrals in (14): Extending the integration limits to infinity and substituting $E_i(r_i) = \exp(-\alpha_i r_i^2)$, $i = 1, 2$ gives, with (3), (4), and (15),

$$\int_0^1\int_0^1 F_{12}dr_1dr_2 = \frac{1}{2n^2}; \qquad D = \frac{1}{4n^4}. \tag{42}$$

Similarly, substituting $\phi_i = \beta_i \cos(\gamma_i r_i)$ and using (15) and (40),

$$\int_0^1\int_0^1 F_{12}\phi_1 dr_1dr_2 = \frac{\beta_1}{2n^2}\left[1 - \frac{\gamma_1\sqrt{2\alpha_2}}{n}\mathfrak{D}\left(\frac{\gamma_1}{n}\sqrt{\frac{\alpha_2}{2}}\right)\right]. \tag{43}$$

Since $\gamma_1 \gg n$, the Dawson integral may be replaced by the first two terms of the asymptotic expansion,[20]

$$\mathfrak{D}(x) \sim \frac{1}{2x}\left[1 + \sum_{m=1}^{\infty}\frac{1.3\cdots(2m-1)}{(2x^2)^m}\right], \qquad x \gg 1, \tag{44}$$

to give

$$\int_0^1\int_0^1 F_{12}\phi_1 dr_1dr_2 \approx -\frac{\beta_1}{2\alpha_2\gamma_1^2}. \tag{45}$$

Further,

$$\int_0^1 \int_0^1 F_{12}\phi_1\phi_2 dr_1 dr_2 = \beta_1\beta_2 \int_0^1 \exp\,(-\alpha_1 r_1^2) r_1 \cos\,(\gamma_1 r_1) \mathcal{J}(r_1) dr_1, \quad (46)$$

where

$$\mathcal{J}(r_1) = \int_0^\infty \exp\,(-\alpha_2 r_2^2) J_0(n r_1 r_2) r_2 \cos\,(\gamma_2 r_2) dr_2. \quad (47)$$

Suhstituting the integral representation of the Bessel function

$$J_0(x) = \frac{1}{\pi} \int_0^\pi \cos\,(x \sin\theta) d\theta, \quad (48)$$

interchanging orders of integration and expanding the cosine product,

$$\mathcal{J}(r_1) = \frac{1}{2\pi} \int_0^\pi [I(\theta) + I(-\theta)] d\theta, \quad (49)$$

where

$$I(\theta) = \int_0^\infty \exp\,(-\alpha_2 r_2^2) r_2 \cos\,[(\gamma_2 + n r_1 \sin\theta) r_2] dr_2. \quad (50)$$

From (40),

$$I(\theta) = \frac{1}{2\alpha_2}\left[1 - \frac{1}{\sqrt{\alpha_2}}(\gamma_2 + n r_1 \sin\theta) \cdot \mathfrak{D}\left\{\frac{1}{2\sqrt{\alpha_2}}(\gamma_2 + n r_1 \sin\theta)\right\}\right]. \quad (51)$$

Since $\gamma_2 \gg n$, hoth Dawson integrals in (49) may he replaced hy the large argument form (44) to give

$$\mathcal{J}(r_1) \approx I(\theta) \approx -\gamma_2^{-2}. \quad (52)$$

Evaluating (46) hy (40) and using (44),

$$\int_0^1 \int_0^1 F_{12}\phi_1\phi_2 dr_1 dr_2 \approx \frac{\beta_1\beta_2}{\gamma_1^2\gamma_2^2}. \quad (53)$$

Suhstituting for the integrals in (14) and reducing (20) then gives

$$\frac{R}{\Delta T_1 + \Delta T_2} \approx -\frac{\beta_1\beta_2}{\beta_1^2 + \beta_2^2}\frac{8n^2}{\gamma_1^2\gamma_2^2}. \quad (54)$$

But $|\beta_1\beta_2/(\beta_1^2 + \beta_2^2)| \leqq \frac{1}{2}$, i.e.,

$$\left|\frac{R}{\Delta T_1 + \Delta T_2}\right| \leqq \left(\frac{2n}{\gamma_1\gamma_2}\right)^2. \quad (55)$$

Since $\gamma_1\gamma_2 \gg 2n$, (55) is much less than unity and so, from (12),

$$\Delta T_{12} \approx \Delta T_1 + \Delta T_2. \quad (56)$$

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